Rules of Inference

Rules of inference provide the justification of the steps used in a proof.

One important rule is called **modus ponens** or the **law of detachment**. It is based on the tautology $(p \land (p \rightarrow q)) \rightarrow q$. We write it in the following way:

$$\begin{array}{l} p \\ p \rightarrow q \\ \hline \therefore q \end{array}$$

The two **hypotheses** p and $p \rightarrow q$ are written in a column, and the **conclusion** below a bar, where $\therefore$ means "therefore".

Rules of Inference

The general form of a rule of inference is:

$$\begin{array}{l} p_1 \\ p_2 \\ \vdots \\ p_n \\ \hline \therefore\ q \end{array}$$

The rule states that if p_1 **and** p_2 **and** … **and** p_n are all true, then q is true as well.

Each rule is an established tautology of

$$p_1 \wedge p_2 \wedge \ldots \wedge p_n \rightarrow q$$

These rules of inference can be used in any mathematical argument and do not require any proof.

Rules of Inference

$$\frac{p}{\therefore\ p \lor q}$$ Addition

$$\frac{p \land q}{\therefore\ p}$$ Simplification

$$\frac{\begin{array}{l} p \\ q \end{array}}{\therefore\ p \land q}$$ Conjunction

$$\frac{\begin{array}{l} \neg q \\ p \to q \end{array}}{\therefore\ \neg p}$$ Modus tollens

$$\frac{\begin{array}{l} p \to q \\ q \to r \end{array}}{\therefore\ p \to r}$$ Hypothetical syllogism (chaining)

$$\frac{\begin{array}{l} p \lor q \\ \neg p \end{array}}{\therefore\ q}$$ Disjunctive syllogism (resolution)

Arguments

Just like a rule of inference, an **argument** consists of one or more hypotheses (or premises) and a conclusion.

We say that an argument is **valid**, if whenever all its hypotheses are true, its conclusion is also true.

However, if any hypothesis is false, even a valid argument can lead to an incorrect conclusion.

Proof: show that hypotheses $\rightarrow$ conclusion is true using rules of inference

Arguments

Example:

"If 101 is divisible by 3, then 101^2 is divisible by 9. 101 is divisible by 3. Consequently, 101^2 is divisible by 9."

Although the argument is **valid**, its conclusion is **incorrect**, because one of the hypotheses is false ("101 is divisible by 3.").

If in the above argument we replace 101 with 102, we could correctly conclude that 102^2 is divisible by 9.

Arguments

Which rule of inference was used in the last argument?

p: "101 is divisible by 3."

q: "101^2 is divisible by 9."

$$\begin{array}{l} p \\ p \rightarrow q \\ \hline \therefore q \end{array}$$

Modus ponens

Unfortunately, one of the hypotheses (p) is false. Therefore, the conclusion q is incorrect.

Arguments

Another example:

"If it rains today, then we will not have a barbeque today. If we do not have a barbeque today, then we will have a barbeque tomorrow. Therefore, if it rains today, then we will have a barbeque tomorrow."

This is a **valid** argument: If its hypotheses are true, then its conclusion is also true.

Arguments

Let us formalize the previous argument:

p: “It is raining today.”

q: “We will not have a barbecue today.”

r: “We will have a barbecue tomorrow.”

So the argument is of the following form:

$$\begin{array}{l} p \rightarrow q \\ q \rightarrow r \\ \hline \therefore P \rightarrow r \end{array}$$

Hypothetical syllogism

Arguments

Another example:

Gary is either intelligent or a good actor.
If Gary is intelligent, then he can count from 1 to 10.
Gary can only count from 1 to 3.
Therefore, Gary is a good actor.

i: "Gary is intelligent."
a: "Gary is a good actor."
c: "Gary can count from 1 to 10."

Arguments

i: "Gary is intelligent."
a: "Gary is a good actor."
c: "Gary can count from 1 to 10."

Step 1:	$\neg c$	Hypothesis
Step 2:	$i \rightarrow c$	Hypothesis
Step 3:	$\neg i$	Modus tollens Steps 1 & 2
Step 4:	$a \vee i$	Hypothesis
Step 5:	a	Disjunctive Syllogism Steps 3 & 4

Conclusion: **a** ("Gary is a good actor.")

Arguments

Yet another example:

If you listen to me, you will pass CS 320.
You passed CS 320.
Therefore, you have listened to me.

Is this argument valid?

No, it assumes $((p \rightarrow q) \land q) \rightarrow p$.

This statement is not a tautology. It is false if p is false and q is true.

Rules of Inference for Quantified Statements

$$\frac{\forall x\, P(x)}{\therefore P(c) \text{ if } c \in U}$$

Universal instantiation

$$\frac{P(c) \text{ for an arbitrary } c \in U}{\therefore \forall x\, P(x)}$$

Universal generalization

$$\frac{\exists x\, P(x)}{\therefore P(c) \text{ for some element } c \in U}$$

Existential instantiation

$$\frac{P(c) \text{ for some element } c \in U}{\therefore \exists x\, P(x)}$$

Existential generalization